

Fibration theorem for Waldhausen K -theory

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Abstract

The main result of this paper is a new flavour of Waldhausen's fibration theorem for Waldhausen K -theory under a different set of hypothesis. The theorem says that for a small category with cofibrations \mathcal{C} and certain classes of weak equivalences v and w in \mathcal{C} , a sequence of simplicial categories:

$$vS.\mathcal{C}^w \rightarrow vS.\mathcal{C} \rightarrow wS.\mathcal{C}$$

is a fibration sequence up to homotopy.

Introduction

The main purpose of this short note is to give a variant of the generic fibration theorem for Waldhausen K -theory. The theorem is first proven in [Wal85] and improved in [Sch06]. To give more precise information, let \mathcal{C} be a small category with cofibrations in the sense of [Wal85] and $v \subset w$ sets of weak equivalences in \mathcal{C} such that w satisfies the extension axiom. Then the full subcategory \mathcal{C}^w of \mathcal{C} consisting of those of objects x such that the canonical morphism $0 \rightarrow x$ is in w is a subcategory with cofibrations in \mathcal{C} and the inclusion functor $\mathcal{C}^w \hookrightarrow \mathcal{C}$ and the identity functor of \mathcal{C} induce a sequence of simplicial categories:

$$vS.\mathcal{C}^w \rightarrow vS.\mathcal{C} \rightarrow wS.\mathcal{C}.$$

The original theorem in [Wal85] or [Sch06] says that if w is saturated and the pair (\mathcal{C}, w) satisfies the factorization axiom, then the sequence above is a fibration sequence up to homotopy. In this paper we give an another sufficient applicable condition which makes the sequence above a fibration sequence up to homotopy. (See Theorem 2.3.) In sections 3 and 4, we will illustrate examples of fibration sequences of K -theory.

Conventions. We mainly follow the notations in [Wal85]. We say that a class w of weak equivalences in a category of cofibrations is *extensional* if w satisfies the extension axiom. For a pair of small categories \mathcal{X} and \mathcal{Y} , we write $\mathcal{Y}^{\mathcal{X}}$ for the category whose objects are functors from \mathcal{X} to \mathcal{Y} and whose morphisms are natural transformations.

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1 Sets of morphisms in a small category

In this section, let \mathcal{C} and \mathcal{D} be small categories and we write $\text{Mor } \mathcal{C}$ for the set of all morphisms in \mathcal{C} . We mainly study heritability of properties for sets of morphisms in \mathcal{C} by taking a right cofinal subset in 1.4, a pull-back by a functor in 1.6 and simplicial constructions in 1.8. We start by giving a glossary about properties for sets of morphisms.

1.1. Definition. Let S be a set of morphisms in \mathcal{C} containing the identity morphisms of objects in \mathcal{C} . We say that S is a *multiplicative set* if S is closed under finite compositions. We say that S is *strictly multiplicative set* if all isomorphisms in \mathcal{C} are in S and if S is closed under finite compositions.

Suppose that $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ is a composable sequence of morphisms in \mathcal{C} . If whenever two of f , g and gf are in S , then the third morphism is also, we say S is a *saturated set* or *satisfies the saturation axiom*. We regard a multiplicative set of \mathcal{C} as a subcategory of \mathcal{C} .

1.2. Definition. Let S and \mathcal{T} be sets of morphisms in \mathcal{C} . We set

$$\mathcal{T} \circ S := \{fg \in \text{Mor } \mathcal{C}; \bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet \text{ where } g \text{ is in } S \text{ and } f \text{ is in } \mathcal{T}\}.$$

We say that S is *right permutative* with respect to \mathcal{T} if for any morphisms $a: x \rightarrow z$ in S and $b: y \rightarrow z$ in \mathcal{T} , there are an object u in \mathcal{C} and morphisms $a': u \rightarrow y$ in S and $b': u \rightarrow x$ in \mathcal{T} such that $ab' = ba'$. We say that S is *right reversible* with respect to \mathcal{T} if for any morphisms $a, a': x \rightarrow y$ in \mathcal{T} , if there exists a morphism $b: y \rightarrow z$ in S such that $ba = ba'$, then there exists a morphism $c: u \rightarrow x$ in S such that $ac = a'c$. We say that S is *right Ore* with respect to \mathcal{T} if S is right permutative and right reversible with respect to \mathcal{T} . We say that S is *right localizing (in \mathcal{C})* if S is a multiplicative and right Ore set with respect to $\text{Mor } \mathcal{C}$. We say that \mathcal{T} is *right cofinal* in S if $\mathcal{T} \subset S$ and for any morphism $x \rightarrow y$ in S , there is a morphism $z \rightarrow x$ in \mathcal{T} such that the composition $z \rightarrow y$ is also in \mathcal{T} .

1.3. Example. (Set of all isomorphisms). We write $i_{\mathcal{C}}$ or shorty i for the class of all isomorphisms in \mathcal{C} . Then $i_{\mathcal{C}}$ is a saturated, strictly multiplicative, right localizing set in \mathcal{C} .

1.4. Lemma. Let $\mathcal{T} \subset S$ and \mathcal{U} be sets of morphisms in \mathcal{C} . Assume that \mathcal{T} is right cofinal in S .

- (1) If S is right permutative with respect to \mathcal{U} and $\mathcal{U} \circ \mathcal{T} \subset \mathcal{U}$, then \mathcal{T} is also right permutative with respect to \mathcal{U} .
- (2) If S is right reversible with respect to \mathcal{U} , then \mathcal{T} is also right reversible with respect to \mathcal{U} .
- (3) If \mathcal{T} is right permutative with respect to \mathcal{U} and $\mathcal{T} \circ \mathcal{U} \subset \mathcal{U}$, then S is also right permutative with respect to \mathcal{U} .
- (4) Assume that S is saturated and right localizing in \mathcal{C} and \mathcal{T} is a multiplicative set. Then the inclusion functor $j: \mathcal{T} \hookrightarrow S$ is a homotopy equivalence.

Proof. (1) In order to show that \mathcal{T} is right permutative with respect to \mathcal{U} , we must produce the dotted morphisms $s''': e \rightarrow a$ in \mathcal{T} and $f'': e \rightarrow b$ in \mathcal{U} in the commutative diagram below with $f \in \mathcal{U}$ and $s \in \mathcal{T}$:

$$\begin{array}{ccc} e & \xrightarrow{f'' \in \mathcal{U}} & b \\ \downarrow s''' \in \mathcal{T} & & \downarrow s \in \mathcal{T} \\ a & \xrightarrow{f \in \mathcal{U}} & c. \end{array}$$

Then by assumption, there are morphisms $f': d \rightarrow b$ in \mathcal{U} and $s': d \rightarrow a$ in S such that $fs' = sf'$. Since \mathcal{T} is right cofinal in S , there is a morphism $s'': e \rightarrow d$ in \mathcal{T} such that the composition $s's'': e \rightarrow a$ is in \mathcal{T} . By the condition $\mathcal{U} \circ \mathcal{T} \subset \mathcal{U}$, the morphism $f's''$ is in \mathcal{U} . We shall set $s''' := s's''$ and $f'' := f's''$.

(2) In order to show that \mathcal{T} is right reversible with respect to \mathcal{U} , for any morphisms $f, g: a \rightarrow b$ in \mathcal{U} such that there is a morphism $s: b \rightarrow c$ in \mathcal{T} such that $sf = sg$, we must produce a morphism $s''': e \rightarrow a$ in \mathcal{T} such that $fs''' = gs'''$. Then there is a morphism $s': d \rightarrow a$ in S such that $fs' = gs'$. Since \mathcal{T} is right cofinal in S , there is a morphism $s'': e \rightarrow d$ such that the composition $s's'': e \rightarrow a$ is in \mathcal{T} . We shall set $s''' := s's''$.

(3) In order to show that S is right permutative with respect to \mathcal{U} , we must produce the dotted morphisms $s'': d \rightarrow a$ in S and $f'': d \rightarrow b$ in \mathcal{U} in the commutative diagram below with $f \in \mathcal{U}$ and $s \in S$:

$$\begin{array}{ccc} d & \xrightarrow{f'' \in \mathcal{U}} & b \\ \downarrow s'' \in \mathcal{T} & & \downarrow s \in S \\ a & \xrightarrow{f \in \mathcal{U}} & c. \end{array}$$

Then there is a morphism $s': b' \rightarrow b$ in \mathcal{T} such that the composition $ss': b' \rightarrow c$ is in \mathcal{T} by assumption. Then there are morphisms $s'': d \rightarrow a$ in \mathcal{T} and $f': d \rightarrow b'$ in \mathcal{U} such that $ss'f' = fs''$. By assumption the composition $s'f': d \rightarrow b$ is in \mathcal{U} . We shall set $f'' := s'f'$.

(4) Let x be an object in \mathcal{S} . We write j/x for the category whose object is a pair (y, a) of an object y in \mathcal{T} and a morphism $a: y \rightarrow x$ in \mathcal{S} and whose morphism $\alpha: (y, a) \rightarrow (z, b)$ is a morphism $\alpha: y \rightarrow z$ in \mathcal{T} such that $a = b\alpha$. Since the object (x, id_x) is in j/x , the category j/x is a non-empty category.

claim. j/x is a cofiltering category. Namely

- (a) For any objects (y, a) and (z, b) in j/x , there are morphisms $\alpha: (w, c) \rightarrow (y, a)$ and $\beta: (w, c) \rightarrow (z, b)$ in j/x .
- (b) For any morphisms $\alpha, \beta: (y, a) \rightarrow (z, b)$ in j/x , there is a morphism $\gamma: (w, c) \rightarrow (y, a)$ such that $\alpha\gamma = \beta\gamma$.

Proof of claim. (a) Since \mathcal{S} is right permutative with respect to $\text{Mor } \mathcal{C}$, there are morphisms $b': w' \rightarrow y$ and $a': w' \rightarrow z$ such that a' is in \mathcal{S} . Then by the saturated axiom for \mathcal{S} , $ba' = ab'$ and b' are also in \mathcal{S} . By right cofinality of \mathcal{T} in \mathcal{S} , there is a morphism $t: w'' \rightarrow w'$ such that the composition $a't$ is in \mathcal{T} . By right cofinality of \mathcal{T} in \mathcal{S} again, there is a morphism $t': w \rightarrow w''$ such that the composition $b'tt'$ is in \mathcal{T} . Then we set $\alpha := b'tt'$, $\beta := a'tt'$ and $c := aa' = b\beta$.

(b) Since \mathcal{S} is right reversible with respect to $\text{Mor } \mathcal{C}$, the equalities $b\alpha = a = b\beta$ implies that there is a morphism $t: w' \rightarrow y$ in \mathcal{S} such that $\alpha t = \beta t$. By right cofinality of \mathcal{T} in \mathcal{S} , there is a morphism $t': w \rightarrow w'$ in \mathcal{T} such that the composition tt' is in \mathcal{T} . We set $\gamma := tt'$ and $c := a\gamma$. \square

By Corollary 2 and Theorem A in [Qui73, §1], we obtain the desired result. \square

1.5. Definition. Let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a functor and \mathcal{S} a non-empty set of morphisms in \mathcal{D} . We define the set of morphisms $\phi^{-1}\mathcal{S}$ in \mathcal{C} the *pull-back of \mathcal{S} by ϕ* by the formula

$$\phi^{-1}\mathcal{S} := \{f \in \text{Mor } \mathcal{C}; \phi(f) \in \mathcal{S}\}.$$

1.6. Lemma. Let $\phi: \mathcal{C} \rightarrow \mathcal{D}$ be a functor and \mathcal{S} a non-empty set of morphisms in \mathcal{D} . Then

- (1) If \mathcal{S} is a multiplicative in \mathcal{C} , then $\phi^{-1}\mathcal{S}$ is also. If \mathcal{S} is a strictly multiplicative in \mathcal{C} , then $\phi^{-1}\mathcal{S}$ is also. If \mathcal{S} is a saturated set in \mathcal{C} , then $\phi^{-1}\mathcal{S}$ is also.
- (2) If ϕ is full and essentially surjective and if \mathcal{T} is right cofinal in \mathcal{S} , then $\phi^{-1}\mathcal{T}$ is right cofinal in $\phi^{-1}\mathcal{S}$.
- (3) If ϕ is an equivalence of categories, \mathcal{S} and \mathcal{T} are strictly multiplicative sets and \mathcal{S} is right permutative with respect to \mathcal{T} , then $\phi^{-1}\mathcal{S}$ is right permutative with respect to $\phi^{-1}\mathcal{T}$.
- (4) If ϕ is an equivalence of categories, \mathcal{S} is a strictly multiplicative and right reversible set with respect to \mathcal{T} , then $\phi^{-1}\mathcal{S}$ is right reversible with respect to $\phi^{-1}\mathcal{T}$.

Proof. (1) Since a functor sends an identity morphism to an identity morphism and an isomorphism to an isomorphism, if \mathcal{S} is closed under identity morphisms, then $\phi^{-1}\mathcal{S}$ is also and if \mathcal{S} is closed under isomorphisms, then $\phi^{-1}\mathcal{S}$ is also. Let $x \xrightarrow{a} y \xrightarrow{b} z$ be a pair of composable morphisms in \mathcal{C} . If two of ba , b and a are in $\phi^{-1}\mathcal{S}$, then two of $\phi(b)\phi(a)$, $\phi(b)$ and $\phi(a)$ are in \mathcal{S} . Therefore if \mathcal{S} is closed under compositions, then $\phi^{-1}\mathcal{S}$ is also and if \mathcal{S} is a saturated set, then $\phi^{-1}\mathcal{S}$ is also.

(2) In order to show that $\phi^{-1}\mathcal{T}$ is right cofinal in $\phi^{-1}\mathcal{S}$, for any morphism $s: x \rightarrow y$ in $\phi^{-1}\mathcal{S}$, we must produce a morphism $t'': z \rightarrow x$ in $\phi^{-1}\mathcal{T}$ such that st'' is also in $\phi^{-1}\mathcal{T}$. Then by assumption, there is a morphism $t: z' \rightarrow \phi(x)$ in \mathcal{T} such that the composition $\phi(s)t$ is in \mathcal{T} . By essential surjectivity of ϕ , there are an object z in \mathcal{C} and an isomorphism $t': \phi(z) \xrightarrow{\sim} z'$. By fullness of ϕ , there is a morphism $t'': z \rightarrow x$ in \mathcal{C} such that $\phi(t'') = tt'$. Since \mathcal{T} is a strictly multiplicative set, the composition tt' is in \mathcal{T} and therefore t'' and the composition st'' are in $\phi^{-1}\mathcal{T}$.

(3) In order to show that $\phi^{-1}\mathcal{S}$ is right permutative with respect to $\phi^{-1}\mathcal{T}$, we must produce the dotted morphisms $s'': w \rightarrow y$ in $\phi^{-1}\mathcal{S}$ and $t'': w \rightarrow x$ in $\phi^{-1}\mathcal{T}$ in the commutative diagram below

with $s \in \phi^{-1} \mathcal{S}$ and $t \in \phi^{-1} \mathcal{T}$:

$$\begin{array}{ccc}
 & s'' \in \phi^{-1} \mathcal{S} & \\
 w & \dashrightarrow & y \\
 \downarrow t'' \in \phi^{-1} \mathcal{T} & & \downarrow t \in \phi^{-1} \mathcal{T} \\
 x & \xrightarrow{s \in \phi^{-1} \mathcal{S}} & z.
 \end{array}$$

Then there are morphisms $s': w' \rightarrow \phi(y)$ in \mathcal{S} and $t': w' \rightarrow \phi(x)$ in \mathcal{T} such that $\phi(s)t' = \phi(t)s'$. By essential surjectivity of ϕ , there are an object w in \mathcal{C} and an isomorphism $u: \phi(w) \xrightarrow{\sim} w'$ in \mathcal{D} . Since \mathcal{S} and \mathcal{T} are strictly multiplicative sets, $s'u$ and $t'u$ are in \mathcal{S} and \mathcal{T} respectively. By fullness of ϕ , there are morphisms $s'': w \rightarrow y$ and $t'': w \rightarrow x$ in \mathcal{C} such that $s'u = \phi(s'')$ and $t'u = \phi(t'')$. Notice that s'' and t'' are in $\phi^{-1} \mathcal{S}$ and $\phi^{-1} \mathcal{T}$ respectively. The equality $\phi(st'') = \phi(s)t'u = \phi(t)s'u = \phi(ts'')$ and faithfulness of ϕ imply the equality $st'' = ts''$.

(4) In order to show that $\phi^{-1} \mathcal{S}$ is right reversible with respect to $\phi^{-1} \mathcal{T}$, for any morphisms $a, b: x \rightarrow y$ in $\phi^{-1} \mathcal{T}$ such that there is a morphism $s: y \rightarrow z$ in $\phi^{-1} \mathcal{S}$ such that $sa = sb$, we must produce a morphism $t': w \rightarrow x$ in $\phi^{-1} \mathcal{S}$ such that $at' = bt'$. Then there is a morphism $t: w' \rightarrow \phi(x)$ in \mathcal{S} such that $at = bt$. By essential surjectivity of ϕ , there are an object in \mathcal{C} and an isomorphism $u: \phi(w) \xrightarrow{\sim} w'$ in \mathcal{D} . Since \mathcal{S} is a strictly multiplicative set, tu is in \mathcal{S} . By fullness of ϕ , there is a morphism $t': w \rightarrow x$ in \mathcal{C} such that $\phi(t') = tu$. Notice that t' is in $\phi^{-1} \mathcal{S}$. The equalities $\phi(at') = \phi(a)tu = \phi(b)tu = \phi(bt')$ and faithfulness of ϕ imply the equality $at' = bt'$. \square

1.7. Definition. For a multiplicative set \mathcal{S} of \mathcal{C} , we define $\mathcal{C}(-, \mathcal{S})$ to be a simplicial subcategory of $\mathcal{C}^{[-]}$ by sending a totally ordered set $[m]$ to $\mathcal{C}(m, \mathcal{S})$ where $\mathcal{C}(m, \mathcal{S})$ is the full subcategory of $\mathcal{C}^{[m]}$ consisting of those functors $x: [m] \rightarrow \mathcal{C}$ such that for any morphism $i \leq j$ in $[m]$, $x(i \leq j)$ is in \mathcal{S} . For each m , we denote an object x in $\mathcal{C}(m, \mathcal{S})$ by

$$x.: x_0 \xrightarrow{i_0^x} x_1 \xrightarrow{i_1^x} x_2 \xrightarrow{i_2^x} \cdots \xrightarrow{i_{m-1}^x} x_m.$$

For a set \mathcal{T} of morphisms in \mathcal{C} , we define $\mathcal{TC}(m, \mathcal{S})$ to be the set of morphisms in $\mathcal{C}(m, \mathcal{S})$ by the formula

$$\mathcal{TC}(m, \mathcal{S}): = \{f \in \text{Mor } \mathcal{C}(m, \mathcal{S}); f_i \text{ is in } \mathcal{T} \text{ for any } 0 \leq i \leq m\}.$$

1.8. Lemma. Let $\mathcal{S}, \mathcal{T}, \mathcal{U}$ and \mathcal{V} be non-empty sets of morphisms in \mathcal{C} and n a non-negative integer. Then

- (1) Assume that \mathcal{T} is a multiplicative set and right cofinal in \mathcal{S} , \mathcal{S} is right permutative with respect to \mathcal{U} and $\mathcal{U} \circ \mathcal{T} \subset \mathcal{T} \circ \mathcal{U}$. Then $\mathcal{TC}(n, \mathcal{U})$ is right cofinal in $\mathcal{SC}(n, \mathcal{U})$.
- (2) Assume that $\mathcal{T} \circ \mathcal{S} \subset \mathcal{T}$ and $\mathcal{U} \circ \mathcal{S} \subset \mathcal{S} \circ \mathcal{U}$, \mathcal{S} is right permutative with respect to both \mathcal{T} and \mathcal{U} and all morphisms in \mathcal{S} are monomorphisms. Then $\mathcal{SC}(n, \mathcal{U})$ is right permutative with respect to $\mathcal{TC}(n, \mathcal{U})$.
- (3) Assume that \mathcal{V} is a multiplicative, right cofinal set in \mathcal{S} , $\mathcal{U} \circ \mathcal{V} \subset \mathcal{V} \circ \mathcal{U}$ and \mathcal{S} is a multiplicative, right permutative with respect to \mathcal{U} and reversible set with respect to \mathcal{T} . Then $\mathcal{SC}(n, \mathcal{U})$ is right reversible with respect to $\mathcal{TC}(n, \mathcal{U})$.

Proof. We proceed by induction on n . If $n = 0$, then the assertion is hypothesis. We assume that $n \geq 1$. Let $\iota: [n-1] \hookrightarrow [n]$ be the inclusion functor. We write $\iota^*: \mathcal{C}(n, \mathcal{U}) \rightarrow \mathcal{C}(n-1, \mathcal{C})$ for the induced functor by the composition with ι .

- (1) In order to show that $\mathcal{TC}(n, \mathcal{U})$ is right cofinal in $\mathcal{SC}(n, \mathcal{U})$, for any morphism $s: a \rightarrow b$ in $\mathcal{SC}(n, \mathcal{U})$, we must produce a morphism $t: c \rightarrow a$ in $\mathcal{TC}(n, \mathcal{U})$ such that st is also in $\mathcal{TC}(n, \mathcal{U})$. Then by assumption, there is a morphism $t': c' \rightarrow a_n$ in \mathcal{T} such that $s_n t': c' \rightarrow b_n$ is in \mathcal{T} . Since \mathcal{T} is right cofinal in \mathcal{S} , the morphism $t': c' \rightarrow a_n$ is in \mathcal{S} . We construct the dotted morphisms $t_k'': c_k' \rightarrow a_k$ ($0 \leq k \leq n-1$) in \mathcal{S} in the commutative diagram below by using the assumption that \mathcal{S} is right

permutative with respect to \mathcal{U} and descending induction on k .

$$\begin{array}{ccccccc}
c''_0 & \xrightarrow{i''_0} & c''_1 & \xrightarrow{i''_1} & \cdots & \xrightarrow{i''_{n-2}} & c''_{n-1} \xrightarrow{i''_{n-1}} c''_n = c' \\
\downarrow t''_0 & & \downarrow t''_1 & & & & \downarrow t''_{n-1} \\
a_0 & \xrightarrow{i^a_0} & a_1 & \xrightarrow{i^a_1} & \cdots & \xrightarrow{i^a_{n-2}} & a_{n-1} \xrightarrow{i^a_{n-1}} a_n.
\end{array}$$

Hence there is the morphism $t'' : c'' \rightarrow a$ in $\mathcal{SC}(n, \mathcal{U})$ such that $c''_n = c'$ and $t''_n = t' : c' \rightarrow a_n$. By applying the inductive hypothesis to $\iota^*(st'') : \iota^*c'' \rightarrow \iota^*b$, there is a morphism $t^{(3)} : c^{(3)} \rightarrow \iota^*c$ in $\mathcal{TC}(n-1, \mathcal{U})$ such that the composition $(\iota^*(st''))t^{(3)}$ is also in $\mathcal{TC}(n-1, \mathcal{U})$. Then by the condition $\mathcal{U} \circ \mathcal{T} \subset \mathcal{T} \circ \mathcal{U}$, there are the dotted morphisms $j : c_{n-1} \rightarrow c^{(4)}$ in \mathcal{U} and $t^{(4)} : c^{(4)} \rightarrow c' = c''_n$ in \mathcal{T} in the commutative diagram below:

$$\begin{array}{ccc}
c_{n-1} & \xrightarrow{j \in \mathcal{U}} & c^{(4)} \\
\downarrow t^{(3)}_{n-1} \in \mathcal{T} & & \downarrow t^{(4)} \in \mathcal{T} \\
c''_{n-1} & \xrightarrow{i^{c''}_{n-1} \in \mathcal{U}} & c' = c''_n.
\end{array}$$

We define $t : c \rightarrow a$ to be a morphism in $\mathcal{TC}(n; \mathcal{U})$ by setting as follows.

$$c_k := \begin{cases} c_k^{(3)} & \text{if } 0 \leq k \leq n-1 \\ c^{(4)} & \text{if } k = n \end{cases}, i_k^c := \begin{cases} i_k^{(3)} & \text{if } 0 \leq k \leq n-2 \\ j & \text{if } k = n-1 \end{cases}, t_k := \begin{cases} t_k^{(3)} & \text{if } 0 \leq k \leq n-1 \\ t_n^{(4)} & \text{if } k = n. \end{cases}$$

Then we can easily check that $st : c \rightarrow b$ is in $\mathcal{TC}(n, \mathcal{U})$.

(2) In order to show that $\mathcal{SC}(n, \mathcal{U})$ is right permutative with respect to $\mathcal{TC}(n, \mathcal{U})$, we must produce the dotted morphisms $p : d \rightarrow a$ in $\mathcal{SC}(n, \mathcal{U})$ and $q : d \rightarrow c$ in $\mathcal{TC}(n, \mathcal{U})$ in the commutative diagram below with $s \in \mathcal{SC}(n, \mathcal{U})$ and $t \in \mathcal{TC}(n, \mathcal{U})$:

$$\begin{array}{ccc}
d & \xrightarrow{q \in \mathcal{TC}(n, \mathcal{U})} & c \\
\downarrow p \in \mathcal{SC}(n, \mathcal{U}) & & \downarrow s \in \mathcal{SC}(n, \mathcal{U}) \\
a & \xrightarrow[t \in \mathcal{TC}(n, \mathcal{U})]{} & b.
\end{array}$$

Then there are morphisms $p' : d' \rightarrow a_n$ in \mathcal{S} and $q' : d' \rightarrow c_n$ in \mathcal{T} such that $s_n q' = t_n p'$. We construct the dotted morphisms $p''_k : d''_k \rightarrow a_k$ ($0 \leq k \leq n-1$) in \mathcal{S} in the commutative diagram below by using the assumption that \mathcal{S} is right permutative with respect to \mathcal{U} and descending induction on k :

$$\begin{array}{ccccccc}
d''_0 & \xrightarrow{i''_0} & d''_1 & \xrightarrow{i''_1} & \cdots & \xrightarrow{i''_{n-2}} & d''_{n-1} \xrightarrow{i''_{n-1}} d''_n = d' \\
\downarrow p''_0 & & \downarrow p''_1 & & & & \downarrow p''_{n-1} \\
a_0 & \xrightarrow{i^a_0} & a_1 & \xrightarrow{i^a_1} & \cdots & \xrightarrow{i^a_{n-2}} & a_{n-1} \xrightarrow{i^a_{n-1}} a_n.
\end{array}$$

Hence we have the morphism $p'' : d'' \rightarrow a$ in $\mathcal{SC}(n, \mathcal{U})$ such that $d''_n = d'$ and $p''_n = p'$. By applying the inductive hypothesis to $\iota^*(tp'') : \iota^*d'' \rightarrow \iota^*b$ in $\mathcal{TC}(n-1, \mathcal{U})$ and $\iota^*s : \iota^*c \rightarrow \iota^*b$ in $\mathcal{SC}(n-1, \mathcal{U})$, we get the dotted morphisms $p^{(3)} : d^{(3)} \rightarrow \iota^*d''$ in $\mathcal{SC}(n-1, \mathcal{U})$ and $q'' : d^{(3)} \rightarrow \iota^*c$ in $\mathcal{TC}(n-1, \mathcal{U})$

in the commutative diagram below:

$$\begin{array}{ccc}
 d^{(3)} & \xrightarrow{q'' \in \mathcal{TC}(n-1, \mathcal{U})} & \iota^* c \\
 \downarrow p^{(3)} \in \mathcal{SC}(n-1, \mathcal{U}) & & \downarrow \iota^* s \in \mathcal{SC}(n-1, \mathcal{U}) \\
 \iota^* d'' & \xrightarrow{\iota^*(tp'')} & \iota^* b.
 \end{array}$$

By assumption $\mathcal{U} \circ \mathcal{S} \subset \mathcal{S} \circ \mathcal{U}$, there are morphisms $j: d_{n-1}^{(3)} \rightarrow d^{(4)}$ in \mathcal{U} and $p^{(4)}: d^{(4)} \rightarrow d' (= d''_n)$ in \mathcal{S} such that $p^{(4)}j = i_{n-1}^{d''}p_{n-1}^{(3)}$. We define $p: d \rightarrow a$ and $q: d \rightarrow c$ to be morphisms in $\mathcal{C}(n, \mathcal{U})$ by setting as follows.

$$d_k := \begin{cases} d_k^{(3)} & \text{if } 0 \leq k \leq n-1 \\ d^{(4)} & \text{if } k = n \end{cases}, \quad i_k^d := \begin{cases} i_k^{d^{(3)}} & \text{if } 0 \leq k \leq n-2 \\ j & \text{if } k = n-1 \end{cases},$$

$$p_k := \begin{cases} p''_k p_k^{(3)} & \text{if } 0 \leq k \leq n-1 \\ p' p^{(4)} & \text{if } k = n \end{cases}, \quad q_k := \begin{cases} q''_k & \text{if } 0 \leq k \leq n-1 \\ q' p^{(4)} & \text{if } k = n. \end{cases}$$

Notice that s_n is a monomorphism by assumption and we have equalities

$$\begin{aligned}
 s_n q_n i_{n-1}^d &= s_n q' p^{(4)} i_{n-1}^d = t_n p' i_{n-1}^{d''} p_{n-1}^{(3)} = t_n i_{n-1}^a p''_{n-1} p_{n-1}^{(3)} \\
 &= i_{n-1}^b t_{n-1} p''_{n-1} p_{n-1}^{(3)} = i_{n-1}^b s_{n-1} q_{n-1} = s_n i_{n-1}^c q_{n-1}.
 \end{aligned}$$

$$\begin{array}{ccccc}
 d_{n-1} & \xrightarrow{q_{n-1}} & c_{n-1} & & \\
 \downarrow p_{n-1}^{(3)} & & \downarrow s_{n-1} & & \\
 d''_{n-1} & \xrightarrow{p''_{n-1}} & a_{n-1} & \xrightarrow{t_{n-1}} & b_{n-1} \\
 \downarrow i_{n-1}^{d''} & & \downarrow i_{n-1}^a & & \downarrow i_{n-1}^b \\
 d' & \xrightarrow{p'} & a_n & \xrightarrow{t_n} & b_n \\
 \downarrow p^{(4)} & & \downarrow q' & & \downarrow s_n \\
 d_n & \xrightarrow{q_n} & c_n & &
 \end{array}$$

Therefore we have the equality $q_n i_{n-1}^d = i_{n-1}^c q_{n-1}$. Namely $q: d \rightarrow c$ is actually a morphism in $\mathcal{C}(n, \mathcal{U})$. By definition, p is in $\mathcal{SC}(n, \mathcal{U})$ and q is in $\mathcal{TC}(n, \mathcal{U})$ and we have $sq = tp$.

(3) In order to show that $\mathcal{SC}(n, \mathcal{U})$ is right reversible with respect to $\mathcal{TC}(n, \mathcal{U})$, for any morphisms $f, g: a \rightarrow b$ in $\mathcal{TC}(n, \mathcal{U})$ such that there is a morphism $s: b \rightarrow c$ in $\mathcal{SC}(n, \mathcal{U})$ such that $sf = sg$, we must produce a morphism $t: d \rightarrow a$ in $\mathcal{SC}(n, \mathcal{U})$ such that $ft = gt$. Then there is a morphism $t': d' \rightarrow a_n$ in \mathcal{S} such that $f_n t' = g_n t'$ by assumption. We construct the dotted morphisms $d''_k: d''_k \rightarrow a_k$ ($0 \leq k \leq n-1$) in \mathcal{S} in the commutative diagram below by using the assumption that \mathcal{S} is right permutative with respect to \mathcal{U} and descending induction on k :

$$\begin{array}{ccccccc}
 d''_0 & \xrightarrow{i_0^{d''}} & d''_1 & \xrightarrow{i_1^{d''}} & \cdots & \xrightarrow{i_{n-2}^{d''}} & d''_{n-1} \xrightarrow{i_{n-1}^{d''}} d''_n = d' \\
 \downarrow t''_0 & & \downarrow t''_1 & & & & \downarrow t''_{n-1} \\
 a_0 & \xrightarrow{i_0^a} & a_1 & \xrightarrow{i_1^a} & \cdots & \xrightarrow{i_{n-2}^a} & a_{n-1} \xrightarrow{i_{n-1}^a} a_n
 \end{array}$$

Hence there is a morphism $t'': d' \rightarrow a$ in $\mathcal{SC}(n, \mathcal{U})$ such that $d''_n = d'$ and $t''_n = t'$. By applying the inductive hypothesis to $\iota^*(ft'')$, $\iota^*(gt''): \iota^*d'' \rightarrow \iota^*b$ and $\iota^*s: \iota^*b \rightarrow \iota^*c$, there is a morphism $t^{(3)}: d^{(3)} \rightarrow \iota^*d''$ in $\mathcal{SC}(n-1, \mathcal{U})$ such that $\iota^*(ft'')t^{(3)} = \iota^*(st'')t^{(3)}$. Since $\mathcal{VC}(n-1, \mathcal{U})$ is a right cofinal set in $\mathcal{SC}(n-1, \mathcal{U})$ by (1), there is a morphism $t^{(4)}: d^{(4)} \rightarrow d^{(3)}$ in $\mathcal{VC}(n-1, \mathcal{U})$ such that the composition $t^{(3)}t^{(4)}$ is in $\mathcal{VC}(n-1, \mathcal{U})$. Then by the condition $\mathcal{U} \circ \mathcal{V} \subset \mathcal{V} \circ \mathcal{U}$, there are morphisms $j: d^{(4)}_{n-1} \rightarrow d^{(5)}$ in \mathcal{U} and $t^{(5)}: d^{(5)} \rightarrow d'$ in \mathcal{S} such that $t^{(5)}j = i^{d'}_{n-1}t^{(3)}_{n-1}t^{(4)}_{n-1}$. We define $t: d \rightarrow a$ to be a morphism in $\mathcal{SC}(n; \mathcal{U})$ by setting as follows.

$$d_k := \begin{cases} d^{(4)}_k & \text{if } 0 \leq k \leq n-1 \\ d^{(5)} & \text{if } k = n \end{cases}, i_k^d := \begin{cases} i^{d^{(4)}}_k & \text{if } 0 \leq k \leq n-2 \\ j & \text{if } k = n-1 \end{cases}, t_k := \begin{cases} t''_k t^{(3)}_k t^{(4)}_k & \text{if } 0 \leq k \leq n-1 \\ t' t^{(5)} & \text{if } k = n. \end{cases}$$

We can easily check that $ft = gt$. □

2 Fibration theorem revisited

In this section, let \mathcal{C} be a small category with cofibrations and we write 0 and $\text{Cof } \mathcal{C}$ for the specific zero object and the set of all cofibrations in \mathcal{C} respectively. For any set of morphisms u in \mathcal{C} , we write \mathcal{C}^u for the full subcategory of those objects x such that the canonical morphism $0 \rightarrow x$ is in u . If a set of weak equivalences w in \mathcal{C} is extensional, then \mathcal{C}^w is a subcategory with cofibrations in \mathcal{C} . For any set of weak equivalences w in \mathcal{C} , we set $\bar{w} := w \cap \text{Cof } \mathcal{C}$. Then the set \bar{w} is strictly multiplicative. For any pair of sets of weak equivalences $v \subset w$ in \mathcal{C} such that w satisfies the extension axiom, the inclusion functor $\mathcal{C}^w \hookrightarrow \mathcal{C}$ and the identity functor of \mathcal{C} induce the sequence

$$vS.\mathcal{C}^w \rightarrow vS.\mathcal{C} \rightarrow wS.\mathcal{C}. \quad (1)$$

The main objective of this section, we will give a sufficient condition that the sequence (1) is a fibration up to homotopy. We start by looking into the proof of the generic fibration theorem in [Wal85].

2.1. Proposition. *Let $v \subset w$ be sets of weak equivalences in \mathcal{C} . Assume that w is extensional. Then the sequence (1) is a fibration up to homotopy if and only if the inclusion functor $\bar{w}S.\mathcal{C}(-, v) \hookrightarrow wS.\mathcal{C}(-, v)$ of bisimplicial categories is a homotopy equivalence.*

Proof. Since w is extensional, we have an isomorphism of bisimplicial categories

$$vS.S.(\mathcal{C}^w \hookrightarrow \mathcal{C}) \xrightarrow{\sim} \bar{w}S.\mathcal{C}(-, v).$$

(See [Wal85, p.352].) Let us consider the following commutative diagram:

$$\begin{array}{ccccc} vS.\mathcal{C}^w & \longrightarrow & vS.\mathcal{C} & \longrightarrow & vS.S.(\mathcal{C}^w \hookrightarrow \mathcal{C}) \\ \parallel & & \parallel & & \downarrow \wr \\ & & & & \bar{w}S.\mathcal{C}(-, v) \\ & & & & \downarrow \text{I} \\ & & & & wS.\mathcal{C}(-, v) \\ & & & & \uparrow \text{II} \\ vS.\mathcal{C}^w & \longrightarrow & vS.\mathcal{C} & \longrightarrow & wS.\mathcal{C}. \end{array}$$

Here the top line is a fibration sequence, up to homotopy and the map II is a homotopy equivalence by [Wal85, 1.5.7., 1.6.5.]. Hence the bottom line is a fibration sequence up to homotopy if and only if the map I is a homotopy equivalence by Lemma 2.2 below. □

2.2. Lemma. *Let*

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ a \downarrow & & b \downarrow & & \downarrow c \\ x' & \longrightarrow & y' & \longrightarrow & z' \end{array} \quad (2)$$

be a map of fibration sequences of topological spaces such that z and z' are connected. If a and b are weak equivalences, then c is also a weak equivalence.

Proof. The diagram (2) above induces a map of distinguished triangles in the stable category \mathbf{Sp} of spectra. Then a and b induces isomorphisms in \mathbf{Sp} and thus c is also an isomorphism in \mathbf{Sp} by the five lemma for distinguished triangles. Since z and z' are connected, c is a weak equivalence. \square

2.3. Theorem. (Fibration theorem). *Let $v \subset w$ be sets of weak equivalences in \mathcal{C} such that w is saturated, extensional and right localizing in \mathcal{C} and right permutative with respect to v and \bar{w} is right cofinal in w and right permutative with respect to both v and $\text{Cof } \mathcal{C}$ and $v \circ \bar{w} \subset \bar{w} \circ v$ and assume that all cofibrations in \mathcal{C} are monomorphisms. Then the sequence (1) is a fibration up to homotopy.*

2.4. Remark. Let w be the set of weak equivalences w in \mathcal{C} and $v = i_{\mathcal{C}}$ the set of all isomorphisms in \mathcal{C} . Then we have $v \circ \bar{w} \subset \bar{w} \circ v$. Namley $v = i_{\mathcal{C}}$ always satisfies the assumption in Theorem 2.3.

2.5. Remark. Theorem 2.3 does not cover the original fibration theorem in [Wal85]. For example let \mathcal{C} be the abelian category of bounded complexes over abelian groups and w is the class of all quasi-isomorphisms in \mathcal{C} . Then Waldhausen category (\mathcal{C}, w) does satisfy Waldhausen's conditions. But \bar{w} is not right cofinal in w as follows. Let $y = \mathbb{Z}/2$ concentrated in degree 0, x a projective resolution of y , say $2: \mathbb{Z} \rightarrow \mathbb{Z}$ and $x \rightarrow y$ the natural projection. The object y has the property that any morphism $z \rightarrow y$ in \bar{w} must be an isomorphism. So y should split off x , and this cannot happen.

2.6. Example. Let \mathcal{E} be a small idempotent complete exact category, \mathcal{A} a right s -filtering subcategory of \mathcal{E} and w a set of all weak isomorphisms associated to \mathcal{A} in \mathcal{E} in the sense of [Sch04]. Then the sets w and $v = i_{\mathcal{E}}$ satisfy assumptions in Theorem 2.3 by [Sch04]. Therefore we have the fibration sequence

$$K(\mathcal{A}) \rightarrow K(\mathcal{E}) \rightarrow K(\mathcal{E}; w).$$

Proof of Theorem 2.3. By proposition 2.1, we shall prove that the inclusion functor $\bar{w}S.\mathcal{C}(-, v) \rightarrow wS.\mathcal{C}(-, v)$ is a homotopy equivalence. To prove $\bar{w}S.\mathcal{C}(-, v) \rightarrow wS.\mathcal{C}(-, v)$ is a homotopy equivalence, the realization lemma in [Seg74, Appendix A] or [Wal78, 5.1] reduces the problem to prove the inclusion functor

$$\bar{w}S_m\mathcal{C}(n, v) \rightarrow wS_m\mathcal{C}(n, v) \quad (*)_{n,m}$$

is a homotopy equivalence for any non-negative integers n and m . To prove the functor $(*)_{n,m}$ is a homotopy equivalence, we will utilize Lemma 1.4 (4) for $\mathcal{C} = S_m\mathcal{C}(n, v)$, $S = w$ and $\mathcal{T} = \bar{w}$. Let n be a non-negative integer and let m be a positive integer. We set $\mathcal{B} := \mathcal{C}(n, v)$ and $\mathcal{A} := \mathcal{B}(m-1, \text{Cof } \mathcal{B})$. Then since the forgetful functor gives an equivalence of categories with cofibrations

$$S_m\mathcal{B} \xrightarrow{\sim} \mathcal{A},$$

to check assumptions in Lemma 1.4 (4), we shall show that $w\mathcal{A}$ is saturated and right localizing in \mathcal{A} and $\bar{w}\mathcal{A}$ is right cofinal in $w\mathcal{A}$ by Lemma 1.6. We enumerate assumptions in Theorem 2.3.

- (A) w is saturated and extensional.
- (B) w is right permutative with respect to $\text{Mor } \mathcal{C}$.
- (C) w is right permutative with respect to v .
- (D) w is right reversible with respect to $\text{Mor } \mathcal{C}$.
- (E) \bar{w} is right cofinal in w .
- (F) \bar{w} is right permutative with respect to $\text{Cof } \mathcal{C}$.
- (G) \bar{w} is right permutative with respect to v .
- (H) $v \circ \bar{w} \subset \bar{w} \circ v$.
- (I) All cofibrations in \mathcal{C} are monomorphisms.

claim. We have the following:

- (1) $w\mathcal{A}$ is extensional and saturated in \mathcal{A} .
- (2) \bar{w} is right permutative with respect to $\text{Mor } \mathcal{C}$.
- (3) $\bar{w}\mathcal{B}$ is right cofinal in $w\mathcal{B}$.
- (4) $\bar{w}\mathcal{B}$ is right permutative with respect to $\text{Mor } \mathcal{B}$.
- (5) $\bar{w}\mathcal{B}$ is right permutative with respect to $\text{Cof } \mathcal{B}$.
- (6) $w\mathcal{B}$ is right permutative with respect to $\text{Cof } \mathcal{B}$.
- (7) $w\mathcal{B}$ is right reversible with respect to $\text{Mor } \mathcal{B}$.
- (8) $\bar{w}\mathcal{A}$ is right cofinal in $w\mathcal{A}$.
- (9) $\bar{w}\mathcal{A}$ is right permutative with respect to $\text{Mor } \mathcal{A}$.
- (10) $w\mathcal{A}$ is right permutative with respect to $\text{Mor } \mathcal{A}$.
- (11) $w\mathcal{A}$ is right reversible with respect to $\text{Mor } \mathcal{A}$.

Proof of claim. Assertion (1) follows from (A) as in [Wal85].

(2) We apply Lemma 1.4 (1) to $\mathcal{S} = w$, $\mathcal{T} = \bar{w}$ and $\mathcal{U} = \text{Mor } \mathcal{C}$. Assumptions in Lemma 1.4 (1) follow from (B), (E) and $\text{Mor } \mathcal{C} \circ \bar{w} \subset \text{Mor } \mathcal{C}$.

(3) We apply Lemma 1.8 (1) to $\mathcal{S} = w$, $\mathcal{T} = \bar{w}$ and $\mathcal{U} = v$. Assumptions in Lemma 1.8 (1) follows from assumptions (E) and (H). Hence we get the result.

(4) We apply Lemma 1.8 (2) to $\mathcal{S} = \bar{w}$, $\mathcal{T} = \text{Mor } \mathcal{C}$ and $\mathcal{U} = v$. Assumptions in Lemma 1.8 (2) follow from previously proved claim (2) and assumptions (C), (H) and (I) and $\text{Mor } \mathcal{C} \circ \bar{w} \subset \text{Mor } \mathcal{C}$. Hence we get the result.

(5) We apply Lemma 1.8 (2) to $\mathcal{S} = \bar{w}$, $\mathcal{T} = \text{Cof } \mathcal{C}$ and $\mathcal{U} = v$. Assumptions in Lemma 1.8 (2) follow from assumptions (F), (G), (H), (I) and $\text{Cof } \mathcal{C} \circ \bar{w} \subset \text{Cof } \mathcal{C}$. Hence we get the result.

(6) We apply Lemma 1.4 (3) to $\mathcal{S} = w\mathcal{B}$, $\mathcal{T} = \bar{w}\mathcal{B}$ and $\mathcal{U} = \text{Cof } \mathcal{B}$. Assumptions in Lemma 1.4 (3) follow from previously proved claims (3) and (5) and $\bar{w}\mathcal{B} \circ \text{Cof } \mathcal{B} \subset \text{Cof } \mathcal{B}$. Hence we obtain the result.

(7) We apply Lemma 1.8 (3) to $\mathcal{S} = \mathcal{V} = w$, $\mathcal{T} = \text{Mor } \mathcal{C}$ and $\mathcal{U} = v$. Assumptions in Lemma 1.8 (3) follow from assumptions (A), (C), (D) and $v \circ w \subset w \circ v$. The last condition follows from assumption $v \subset w$. Hence we get the result.

(8) We apply Lemma 1.8 (1) to $\mathcal{S} = w\mathcal{B}$, $\mathcal{T} = \bar{w}\mathcal{B}$ and $\mathcal{U} = \text{Cof } \mathcal{B}$. Assumptions in Lemma 1.8 (1) follow from previously proved claim (6) and $\bar{w}\mathcal{B} \circ \text{Cof } \mathcal{B} \subset \bar{w}\mathcal{B} \circ \text{Cof } \mathcal{B}$. The last condition follows from $\bar{w}\mathcal{B} \circ \text{Cof } \mathcal{B} \subset \text{Cof } \mathcal{B}$.

(9) We apply Lemma 1.8 (2) to $\mathcal{S} = \bar{w}\mathcal{B}$, $\mathcal{T} = \text{Mor } \mathcal{B}$ and $\mathcal{U} = \text{Cof } \mathcal{B}$. Assumptions in Lemma 1.8 (2) follow from previously proved claim (4) and $\text{Cof } \mathcal{C} \circ \bar{w}\mathcal{B} \subset \bar{w}\mathcal{B} \circ \text{Cof } \mathcal{C}$ and $\text{Mor } \mathcal{B} \circ \bar{w}\mathcal{B} \subset \text{Mor } \mathcal{B}$.

(10) We apply Lemma 1.4 (3) to $\mathcal{S} = w\mathcal{A}$, $\mathcal{T} = \bar{w}\mathcal{A}$ and $\mathcal{U} = \text{Mor } \mathcal{A}$. Assumptions in Lemma 1.4 (3) follow from previously proved claims (8) and (9) and $\bar{w}\mathcal{A} \circ \text{Mor } \mathcal{A} \subset \text{Mor } \mathcal{A}$.

(11) We apply Lemma 1.8 (3) to $\mathcal{S} = w\mathcal{B}$, $\mathcal{T} = \text{Mor } \mathcal{B}$ and $\mathcal{U} = \text{Cof } \mathcal{B}$ and $\mathcal{V} = \bar{w}\mathcal{B}$. Assumptions in Lemma 1.8 (2) follow from previously proved claims (3), (6) and (7) and the facts that $\text{Cof } \mathcal{B} \circ \bar{w}\mathcal{B} \subset \bar{w}\mathcal{B} \circ \text{Cof } \mathcal{B}$ and $\bar{w}\mathcal{B}$ and $w\mathcal{B}$ are multiplicative sets in \mathcal{B} . \square

Hence the inclusion functor between bisimplicial categories

$$\bar{w}S.\mathcal{C}(-, v) \hookrightarrow wS.\mathcal{C}(-, v)$$

is a homotopy equivalence. Then the sequence (1) is a fibration sequence up to homotopy by Proposition 2.1. \square

3 Localization of categories

In this section, we construct fibration sequences of K -theory of categories with cofibrations relating with localizations of categories. (See Corollary 3.5.) A key ingredient of the construction is a homotopy invariance of taking Gabriel-Zisman localization. (See Lemma 3.3.) By utilizing the same method, in the last of this section, we will prove a version of approximation theorem for Waldhausen K -theory. (See Proposition 3.6.) We start by recalling the notations in localization theory of categories from [GZ67].

3.1. Definition-Theorem. (cf. [GZ67, Chapter I].) Let \mathcal{C} be a small category and \mathcal{S} a right localizing set in \mathcal{C} . Then

(1) We define $\mathcal{S}^{-1}\mathcal{C}$ to be a category by setting $\text{Ob } \mathcal{S}^{-1}\mathcal{C} := \text{Ob } \mathcal{C}$ and

$$\text{Hom}_{\mathcal{S}^{-1}\mathcal{C}}(x, y) := \text{colim}_{z \rightarrow x \in \mathcal{S}} \text{Hom}(z, y)$$

for any objects x and y in \mathcal{C} . A morphism from an object x to an object y in $\mathcal{S}^{-1}\mathcal{C}$ is represented by a sequence of morphisms $x \xleftarrow{s} z \xrightarrow{f} y$ in \mathcal{C} with $s \in \mathcal{S}$. We denote it by f/s . We also define $Q_{\mathcal{S}}: \mathcal{C} \rightarrow \mathcal{S}^{-1}\mathcal{C}$ to be a functor by sending an object x to x and a morphism $x \xrightarrow{f} y$ to the class $x \xleftarrow{\text{id}_x} x \xrightarrow{f} y$.

(2) The pair $(\mathcal{S}^{-1}\mathcal{C}, Q_{\mathcal{S}})$ satisfies the following universal property:

For any category \mathcal{X} , the functor

$$\mathcal{X}^{Q_{\mathcal{S}}}: \mathcal{X}^{\mathcal{S}^{-1}\mathcal{C}} \rightarrow \mathcal{X}^{\mathcal{C}}$$

is an isomorphism from $\mathcal{X}^{\mathcal{S}^{-1}\mathcal{C}}$ onto the full subcategory of $\mathcal{X}^{\mathcal{C}}$ whose objects are the functor $f: \mathcal{C} \rightarrow \mathcal{X}$ which makes all the morphisms of \mathcal{S} invertible. We say that $\mathcal{S}^{-1}\mathcal{C}$ is the *category of fractions of \mathcal{C} for \mathcal{S}* , and $Q_{\mathcal{S}}$ is the *canonical functor*. \square

3.2. For a category \mathcal{C} , we write \mathcal{C}^{\times} for the maximum groupoid of \mathcal{C} , namely \mathcal{C}^{\times} is the category whose class of objects are same as \mathcal{C} and whose class of morphisms are all isomorphisms in \mathcal{C} .

3.3. Lemma. Let \mathcal{C} be a small category and w is a right localizing set in \mathcal{C} . Then

(1) The canonical functor $Q_w: \mathcal{C} \rightarrow w^{-1}\mathcal{C}$ is a homotopy equivalence.

(2) Moreover assume that w is saturated in \mathcal{C} , then Q_w induces an equivalence of categories $w^{-1}w \xrightarrow{\sim} (w^{-1}\mathcal{C})^{\times}$.

Proof. (1) We will apply Quillen's Theorem A in [Qui73] to the functor Q_w . Let x be an object in $w^{-1}\mathcal{C}$. We write x/Q_w for the category consisting of pairs (y, α) with $\alpha: x \rightarrow y$ morphisms in $w^{-1}\mathcal{C}$, in which a morphism from (y, α) to (y', α') is a morphism $u: y \rightarrow y'$ such that $Q_w(u)\alpha = \alpha'$. We will show that x/Q_w is cofiltered. Namely we will check the following three conditions:

(a) x/Q_w is non-empty.

(b) For every two objects (y, α) and (y', α') in x/Q_w , there exists an object (y'', α'') and two morphisms $u: (y'', \alpha'') \rightarrow (y, \alpha)$ and $v: (y'', \alpha'') \rightarrow (y', \alpha')$.

(c) For every two parallel morphisms $u, v: (y, \alpha) \rightarrow (y', \alpha')$ in x/Q_w , there exists an object (y'', α'') and a morphism $j: (y'', \alpha'') \rightarrow (y, \alpha)$ such that $uj = vj$.

Since $(x, \text{id}_x / \text{id}_x)$ is in x/Q_w , x/Q_w is non-empty. Let $(y, \alpha/s)$ and $(y', \alpha'/s')$ with $x \xleftarrow{s} z \xrightarrow{\alpha} y$ and $x' \xleftarrow{s'} z' \xrightarrow{\alpha'} y'$ be a pair of objects in x/Q_w . We will show condition (b). By right permutative condition, there exists morphisms $t': z'' \rightarrow z$ and $t: z'' \rightarrow z'$ such that $s't = st'$. Then we can check that $(z'', st' / \text{id}_{z''})$ is in x/Q_w and there exists morphisms $(y', \alpha'/s') \xrightarrow{\alpha't} (z'', st' / \text{id}_{z''}) \xrightarrow{\alpha't'} (y, \alpha/s)$. Next we will show condition (c). For any two parallel morphisms $u, v: (y, \alpha/s) \rightarrow (y', \alpha'/s')$, by using right permutative condition, we can find morphisms $t: z'' \rightarrow z$ and $t': z'' \rightarrow z'$ such that $st = s't'$, $uat = \alpha't' = vat$. Hence there exists a morphism $\alpha t: (z'', st / \text{id}_{z''}) \rightarrow (y, \alpha/s)$ such that $uat = vat$. Therefore x/Q_w is contractible by Corollary 2 (of Proposition 3) in [Qui73, §1]. Thus by Theorem A in [Qui73] again, Q_w is a homotopy equivalence.

(2) Obviously the functor $w^{-1}w \rightarrow (w^{-1}\mathcal{C})^{\times}$ is essentially surjective, what we need to show is that the functor is fully faithful. Let x and y be a pair of objects in \mathcal{C} and $x \xleftarrow{s} z \xrightarrow{f} y$ a morphism in $w^{-1}\mathcal{C}$. We will show that f/s is an isomorphism if and only if f is in w . If f is in w , we can check that s/f is the inverse morphism of f/s . If f/s is an isomorphism, then there exists the inverse morphism $y \xleftarrow{t} z' \xrightarrow{g} x$ of f/s . Since $f/s \cdot g/t = \text{id}_x$, there exists morphisms $s': z'' \rightarrow z'$ in w and $g': z'' \rightarrow z$ such that $sg' = gs'$ and $ts' = fg'$. By saturation condition, it turns out that f is in w . Thus the functor $w^{-1}w \rightarrow (w^{-1}\mathcal{C})^{\times}$ is full. Next let $x \xleftarrow{s} z \xrightarrow{f} y$ and $x' \xleftarrow{s'} z' \xrightarrow{f'} y'$ be a pair of morphisms in $w^{-1}w$. If the equality holds in $w^{-1}\mathcal{C}$. Then there exists morphisms $t: z'' \rightarrow z$ and $t': z'' \rightarrow z'$ such that we have equalities $s't' = st$ and $f't' = ft$ and st is in w . Therefore by saturation condition, t and t' are also in w and the equality $f/s = f'/s'$ holds in $w^{-1}w$. Thus the functor $w^{-1}w \rightarrow (w^{-1}\mathcal{C})^{\times}$ is faithful. Hence we complete the proof. \square

3.4. Corollary. Let (\mathcal{C}, w) be a small Waldhausen category. Assume that w is a saturated, right localizing set and $w^{-1}\mathcal{C}$ is also a category with cofibrations such that the canonical functor $Q_w: \mathcal{C} \rightarrow w^{-1}\mathcal{C}$ is exact and reflects exactness and for any non-negative integer n , Q_w induces an equivalence of categories $w^{-1}S_n\mathcal{C} \xrightarrow{\sim} S_n w^{-1}\mathcal{C}$. Then Q_w induces a homotopy equivalence $wS.\mathcal{C} \rightarrow iS.w^{-1}\mathcal{C}$.

Proof. For any non-negative integer n , let us consider the commutative diagram of the canonical functors

$$\begin{array}{ccc} wS_n\mathcal{C} & \xrightarrow{\quad} & iS_n w^{-1}\mathcal{C} \\ & \searrow \text{I} & \nearrow \text{II} \\ & (w^{-1}S_n\mathcal{C})^\times & \end{array}$$

Here the functor **I** is a homotopy equivalence and the functor **II** is an equivalence of categories by the previous lemma 3.3. Hence the canonical functor induced from Q_w is a homotopy equivalence $wS.\mathcal{C} \rightarrow iS.w^{-1}\mathcal{C}$ by the realization lemma in [Seg74, Appendix A] or [Wal78, 5.1]. \square

3.5. Corollary. Let (\mathcal{C}, w) be a small Waldhausen category. We set $\bar{w} := w \cap \text{Cof } \mathcal{C}$. We assume the following conditions:

- (i) w is saturated.
- (ii) $wS_n\mathcal{C}$ is right localizing in $S_n\mathcal{C}$ for any non-negative integer n .
- (iii) $w^{-1}\mathcal{C}$ is also a category with cofibrations such that the canonical functor $Q_w: \mathcal{C} \rightarrow w^{-1}\mathcal{C}$ is exact and reflects exactness.
- (iv) For any non-negative integer n , Q_w induces an equivalence of categories $w^{-1}S_n\mathcal{C} \xrightarrow{\sim} S_n w^{-1}\mathcal{C}$.
- (v) \bar{w} is right cofinal in w .
- (vi) \bar{w} is right permutative with respect to $\text{Cof } \mathcal{C}$.
- (vii) All cofibrations in \mathcal{C} are monomorphisms.

Then the inclusion functor $\mathcal{C}^w \hookrightarrow \mathcal{C}$ and the canonical localization functor $Q_w: \mathcal{C} \rightarrow w^{-1}\mathcal{C}$ induces a fibration sequence up to homotopy

$$iS.\mathcal{C}^w \rightarrow iS.\mathcal{C} \rightarrow iS.w^{-1}\mathcal{C}.$$

Proof. Let us consider the commutative diagram below:

$$\begin{array}{ccccc} iS.\mathcal{C}^w & \longrightarrow & iS.\mathcal{C} & \longrightarrow & wS.\mathcal{C} \\ & & & \searrow & \downarrow \text{I} \\ & & & iS.Q_w & iS.w^{-1}\mathcal{C}. \end{array}$$

Then the top line is a fibration sequence up to homotopy by Theorem 2.3 and the map **I** is a homotopy equivalence by Corollary 3.4. Hence we obtain the result. \square

By using the similar method as in the proof of Corollary 3.4, we prove a version of an approximation theorem for Waldhausen K -theory in the following way.

3.6. Proposition (Approximation theorem). Let $f: (\mathcal{D}, v) \rightarrow (\mathcal{C}, w)$ be an exact functor between Waldhausen categories. Suppose that for any non-negative integer n , $wS_n\mathcal{C}$ and $vS_n\mathcal{D}$ are right localizing in $S_n\mathcal{C}$ and $S_n\mathcal{D}$ respectively and f induces an equivalence of categories $v^{-1}S_n\mathcal{D} \xrightarrow{\sim} w^{-1}S_n\mathcal{C}$. Then f induces a homotopy equivalence $vS.\mathcal{D} \rightarrow wS.\mathcal{C}$.

Proof. The realization lemma in [Seg74, Appendix A] or [Wal78, 5.1] reduces the problem to prove that for any non-negative integer n , $vS_n\mathcal{D} \rightarrow wS_n\mathcal{C}$ the induced morphism by f is a homotopy equivalence. We fix a non-negative integer n and let us consider the commutative diagram below.

$$\begin{array}{ccc} vS_n\mathcal{D} & \longrightarrow & wS_n\mathcal{C} \\ \downarrow & & \downarrow \\ (v^{-1}S_n\mathcal{D})^\times & \xrightarrow{\sim} & (w^{-1}S_n\mathcal{C})^\times. \end{array}$$

Here the bottom line is an equivalence of categories, a fortiori, a homotopy equivalence and the vertical morphisms are homotopy equivalences by Lemma 3.3. Hence the inclusion functor in $vS_n \mathcal{D} \rightarrow wS_n \mathcal{C}$ is a homotopy equivalence. We complete the proof. \square

4 Localization of diagrams of modules

Let A be a noetherian commutative ring with 1 and \mathfrak{S} a multiplicative closed set of A . Let \mathcal{M}_A be the category of finitely generated A -modules. We denote a skelton of \mathcal{M}_A by the same letters \mathcal{M}_A and we shall assume that \mathcal{M}_A is a small category. In this subsection, we will study the category of diagrams in $\mathcal{M}_{\mathfrak{S}^{-1}A}$.

4.1. Recall from Conventions that for a small category \mathcal{I} and a category \mathcal{X} , we write $\mathcal{X}^{\mathcal{I}}$ for the category of functors from \mathcal{I} to \mathcal{X} whose morphisms are natural transformations. We sometimes call a functor $\mathcal{I} \rightarrow \mathcal{X}$ an \mathcal{I} -*diagram* in \mathcal{X} . For a small category \mathcal{I} , the category $\mathcal{M}_A^{\mathcal{I}}$ is enriched over the category of A -modules. Namely for any objects x and y in $\mathcal{M}_A^{\mathcal{I}}$, $\text{Hom}_{\mathcal{M}_A^{\mathcal{I}}}(x, y)$ naturally becomes A -modules. Indeed, for any $f, g: x \rightarrow y$ in $\text{Hom}_{\mathcal{M}_A^{\mathcal{I}}}(x, y)$ and a in A , we set

$$(f + g)_i(t) := f_i(t) + g_i(t),$$

$$(af)_i(t) := af_i(t)$$

for any object i of \mathcal{I} and any element t of x . Moreover the A -modules structures on Hom-sets are compatible with the composition of morphisms. $\mathcal{M}_A^{\mathcal{I}}$ is also an abelian category.

4.2. Let \mathcal{I} be a small category. The base change functor $- \otimes_A \mathfrak{S}^{-1}A: \mathcal{M}_A \rightarrow \mathcal{M}_{\mathfrak{S}^{-1}A}$ which sends an A -module M to $\mathfrak{S}^{-1}M$ induces an exact functor

$$\mathcal{L}_{\mathcal{I}, A, \mathfrak{S}}: \mathcal{M}_A^{\mathcal{I}} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{I}}.$$

For simplicity, we sometimes write \mathcal{L} for $\mathcal{L}_{\mathcal{I}, A, \mathfrak{S}}$.

In the rest of this subsection, let $\{\mathcal{I}_i\}_{1 \leq i \leq r}$ be a family of finite totally ordered sets. We set $\mathcal{J} := \prod_{i=1}^r \mathcal{I}_i$.

4.3. Proposition. *For any objects x and y in $\mathcal{M}_A^{\mathcal{J}}$, there is a canonical isomorphisms of $\mathfrak{S}^{-1}A$ -modules*

$$\text{Hom}_{\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}}(\mathcal{L}(x), \mathcal{L}(y)) \xrightarrow{\sim} \mathfrak{S}^{-1} \text{Hom}_{\mathcal{M}_A^{\mathcal{J}}}(x, y). \quad (3)$$

Proof. We proceed by induction on r . If \mathcal{J} is $[0]$ the totally ordered set $\{0\}$, then $\mathcal{M}_A^{\mathcal{J}}$ and $\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$ are \mathcal{M}_A and $\mathcal{M}_{\mathfrak{S}^{-1}A}$ respectively and assertion follows from Proposition 19 in [Bou61, Chapter II §2.7]. We assume that assertion is true for $r = r'$. We will prove assertion for $r = r' + 1$. We set $\mathcal{C} := \mathcal{M}_A^{\prod_{i=1}^{r'} \mathcal{I}_i}$ and $\mathcal{C}' := \mathcal{M}_{\mathfrak{S}^{-1}A}^{\prod_{i=1}^{r'} \mathcal{I}_i}$. Without loss of generality, we shall assume that $\mathcal{I}_{r'+1}$ is $[n]$ the finite totally ordered set of non-negative integres less than n with the usual ordering for some positive integer n .

Then there is a description of Hom-sets of $\mathcal{C}^{[n]}$ in terms of suitable kernels of finite direct sum of Hom-sets in \mathcal{C} as follows. Let x and y be objects in $\mathcal{C}^{[n]}$ and let i be a non-negative integer less than $n - 1$. We define $D_{x,y,i}: \text{Hom}_{\mathcal{C}}(x_i, y_i) \times \text{Hom}_{\mathcal{C}}(x_{i+1}, y_{i+1}) \rightarrow \text{Hom}_{\mathcal{C}}(x_i, y_{i+1})$ to be a homomorphism of A -modules by sending a pair $(a: x_i \rightarrow y_i, b: x_{i+1} \rightarrow y_{i+1})$ to $bx(i \leq i+1) - y(i \leq i+1)a$. Then the family $\{D_{x,y,i}\}_{1 \leq i \leq n-1}$ induces a homomorphism of A -modules

$$D_{x,y}: \bigoplus_{i \in [n]} \text{Hom}_{\mathcal{C}}(x_i, y_i) \rightarrow \bigoplus_{i \in [n-1]} \text{Hom}_{\mathcal{C}}(x_i, y_{i+1})$$

and there is a canonical isomorphism of A -modules

$$\text{Hom}_{\mathcal{C}^{[n]}}(x, y) \xrightarrow{\sim} \text{Ker } D_{x,y}. \quad (4)$$

Since the base change functor $- \otimes_A \mathfrak{S}^{-1} A: \mathcal{M}_A \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}$ is exact, there are isomorphisms of $\mathfrak{S}^{-1} A$ -modules

$$\begin{aligned}
\mathfrak{S}^{-1} \text{Hom}_{\mathcal{C}[n]}(x, y) &\xrightarrow{\sim} \text{Ker} \left(\bigoplus_{i \in [n]} \mathfrak{S}^{-1} \text{Hom}_{\mathcal{C}}(x_i, y_i) \rightarrow \bigoplus_{i \in [n-1]} \mathfrak{S}^{-1} \text{Hom}_{\mathcal{C}}(x_i, y_{i+1}) \right) \\
&\xrightarrow[\text{I}]{\sim} \text{Ker} \left(\bigoplus_{i \in [n]} \text{Hom}_{\mathcal{C}'}(\mathcal{L}(x_i), \mathcal{L}(y_i)) \rightarrow \bigoplus_{i \in [n-1]} \text{Hom}_{\mathcal{C}'}(\mathcal{L}(x_i), \mathcal{L}(y_{i+1})) \right) \\
&\xrightarrow[\text{II}]{\sim} \text{Ker } D_{\mathcal{L}(x), \mathcal{L}(y)} \\
&= \text{Hom}_{\mathcal{C}'[n]}(\mathcal{L}(x), \mathcal{L}(y)).
\end{aligned}$$

Here the isomorphisms **I** and **II** come from inductive hypothesis and the isomorphism (4) for $\mathfrak{S}^{-1} A$ respectively. \square

4.4. Definition-Corollary. We consider the canonical exact functor $\mathcal{L}_*: \mathcal{M}_A^{\mathcal{J}} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}^{\mathcal{J}}$ induced from the base change functor $- \otimes_A \mathfrak{S}^{-1} A: \mathcal{M}_A \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}$. Then

- (1) For morphisms $f, g: x \rightarrow y$ in $\mathcal{M}_A^{\mathcal{J}}$, $\mathcal{L}_*(f) = \mathcal{L}_*(g)$ if and only if there is an element s in \mathfrak{S} such that $sf = sg$.
- (2) Let \mathcal{C} be a full subcategory of $\mathcal{M}_A^{\mathcal{J}}$. We write $w := \text{Isom}_{\mathfrak{S}, \mathcal{C}}$ for the class of all morphisms f in \mathcal{C} such that $\mathcal{L}_*(f)$ are isomorphisms. Then $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$ is a saturated strictly multiplicative set.
- (3) A morphism $f: x \rightarrow y$ in \mathcal{C} is in $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$ if and only if there are a morphism $g: y \rightarrow x$ and elements s, t and u in \mathfrak{S} such that $fgt = st \text{id}_y$ and $gfu = su \text{id}_x$.
- (4) Let $f: x \rightarrow y$ be a morphism in $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$ and $g: z \rightarrow y$ a morphism in \mathcal{C} . Then there are a morphism $h: z \rightarrow x$ and an element s of \mathfrak{S} such that $sg = fh$.
- (5) w is a right localizing system in \mathcal{C} .
- (6) For any pair of objects x and y in \mathcal{C} , a morphism from x to y in $w^{-1} \mathcal{C}$ is represented by $f/s \text{id}_x$ where f is a morphism from x to y and s is an element of \mathfrak{S} .
- (7) The induced functor $w^{-1} \mathcal{C} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}^{\mathcal{J}}$ from \mathcal{L}_* is fully faithful.
- (8) Assume that \mathcal{C} is a category with cofibrations such that the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{M}_A^{\mathcal{J}}$ is exact and reflects exactness, then $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$ satisfies the guling axiom.
- (9) Moreover assume that $w^{-1} \mathcal{C}$ is a category with cofibrations such that the induced functor $w^{-1} \mathcal{C} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}^{\mathcal{J}}$ from \mathcal{L}_* is exact and reflects exactness. Let n be a non-negative integer. We regard $F_n \mathcal{C}$ as a full subcategory of $\mathcal{M}_A^{[n] \times \mathcal{J}}$. Then we have the equality $w F_n \mathcal{C} = \text{Isom}_{\mathfrak{S}, F_n \mathcal{C}}$.
- (10) We assume same assumptions as in (9). Then for any non-negative integer the canonical functor $w^{-1} S_n \mathcal{C} \rightarrow S_n w^{-1} \mathcal{C}$ induced from the functor $Q_w: \mathcal{C} \rightarrow w^{-1} \mathcal{C}$ is fully faithful.
- (11) Let n be a non-negative integer and assume that the condition:

For any object x in $S_n w^{-1} \mathcal{C}$, there are an object y in $S_n \mathcal{C}$ and a morphism $s: y \rightarrow x$ in w .

Then the canonical functor $w^{-1} S_n \mathcal{C} \rightarrow S_n w^{-1} \mathcal{C}$ induced from Q_w is an equivalence of categories.

- (12) If the condition in (11) holds for any non-negative integer n , then Q_w induces a homotopy equivalence $wS \cdot \mathcal{C} \rightarrow iS \cdot w^{-1} \mathcal{C}$ on K -theory.

- (13) Assume that \mathcal{C} is closed under taking kernels of the morphisms in \mathcal{C} which are epimorphisms in $\mathcal{M}_A^{\mathcal{J}}$. Then for any pair of composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in \mathcal{C} , if gf and f are cofibrations in \mathcal{C} , then g is also a cofibration in \mathcal{C} .

- (14) Assume that the condition in (13) and moreover assume that for any element s of \mathfrak{S} and for any object x in \mathcal{C} , $\text{Coim}(s: x \rightarrow x)$ and $\text{Coker}(s: x \rightarrow x)$ are in \mathcal{C} . Then $\bar{w} := w \cap \text{Cof } \mathcal{C}$ is right permutative with respect to $\text{Cof } \mathcal{C}$.

Proof. (1) By Proposition 4.3, the equality $f/1 = g/1$ holds in $\text{Hom}_{\mathcal{M}_{\mathfrak{S}^{-1} A}^{\mathcal{J}}}(\mathcal{L}_*(x), \mathcal{L}_*(y))$ if and only if there is an element s in \mathfrak{S} such that $(f \cdot 1 - g \cdot 1)s = 0$. Thus we obtain the result.

- (2) Let $x \xrightarrow{f} y \xrightarrow{g} z$ be a pair of composable morphisms in $\mathcal{M}_A^{\mathcal{J}}$. If f is an isomorphism, then $\mathcal{L}_*(f)$ is also an isomorphism. Therefore f is in w . Thus w contains all isomorphisms in $\mathcal{M}_A^{\mathcal{J}}$. Next if two of $\mathcal{L}_*(f)$, $\mathcal{L}_*(g)$ and $\mathcal{L}_*(gf)$ are isomorphisms, then the third one is also an isomorphism. Hence w is a saturated set.

(3) If f is in w . Then there is a morphism $g/s: y \rightarrow x$ in $\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$ such that $fg/s = \text{id}_y/1$ and $gf/s = \text{id}_x/1$ in $\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$. These equalities mean that there are elements t and u in \mathfrak{S} such that $(fg - s\text{id}_y)t = 0$ and $(gf - s\text{id}_x)u = 0$. Conversely if there are a morphism $g: y \rightarrow x$ and elements s, t and u in \mathfrak{S} such that $fgt = st\text{id}_y$ and $gfu = su\text{id}_x$, then $f/1 \cdot g/s = \text{id}_y/1$ and $g/s \cdot f/1 = \text{id}_x/1$ in $\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$.

(4) By (3), there are a morphism $h': y \rightarrow x$ in \mathcal{C} and elements s', t and u in \mathfrak{S} such that $fh't = s't\text{id}_y$ and $h'fu = s'u\text{id}_x$. We set $h := h'gt$ and $s := s't$. Then we have equalities $fh = fh'tg = s'tg = sg$.

(5) By (2) and (4), what we need to prove is that w is right permutative. First we will show right permutative condition for w . Namely for any pair of morphisms $f, g: x \rightarrow y$ in \mathcal{C} if there exists a morphism $s: y \rightarrow z$ in w such that $sf = sg$, then we need to produce a morphism $t: u \rightarrow x$ in w such that $ft = gt$. Indeed, in this case we have equalities

$$\mathcal{L}_*(f) = \mathcal{L}(s)^{-1} \mathcal{L}_*(sf) = \mathcal{L}(s)^{-1} \mathcal{L}_*(sg) = \mathcal{L}_*(g).$$

Hence by (1), there is an element t of \mathfrak{S} such that $ft\text{id}_x = tf = tg = gt\text{id}_x$. Thus we complete the proof.

(6) Let x and y be a pair of objects in \mathcal{C} . A morphism from x to y in $w^{-1}\mathcal{C}$ is represented by f/t where $f: z \rightarrow y$ is a morphism in \mathcal{C} and $t: z \rightarrow x$ is a morphism in w by Definition-Theorem 3.1. Then by (3), there is a morphism $g: x \rightarrow z$ and an element s of \mathfrak{S} such that $tg = s\text{id}_x$. Therefore we have the equality $f/t = fg/s\text{id}_x$ in $w^{-1}\mathcal{C}$. Thus we complete the proof.

(7) Let x and y be a pair of objects of \mathcal{C} . By virtue of Proposition 4.3, a morphism $\mathcal{L}_*(x)$ to $\mathcal{L}_*(y)$ in $\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$ is represented by f/s where f is a morphism from x to y in \mathcal{C} and s is an element of \mathfrak{S} . Hence we have the equality $\mathcal{L}_*(f/s\text{id}_x) = f/s$ and the functor $w^{-1}\mathcal{C} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$ is full. Next we consider a pair of morphisms from x to y in $w^{-1}\mathcal{C}$. By (6), they are represented by $f/s\text{id}_x$ and $f'/s'\text{id}_x$ where $f, f': x \rightarrow y$ are morphisms in \mathcal{C} and s and s' are elements in \mathfrak{S} . Assume that we have the equality $\mathcal{L}_*(f/s\text{id}_x) = \mathcal{L}_*(f'/s'\text{id}_x)$. Then there is an element t of \mathfrak{S} such that $ts'f = tsf'$. Then we have the equalities $f/s\text{id}_x = ts'f/sts'\text{id}_x = tsf'/sts'\text{id}_x = f'/s'\text{id}_x$. Thus the functor $w^{-1}\mathcal{C} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$ induced from \mathcal{L}_* is faithful.

(8) Let us consider the commutative diagram in \mathcal{C} below

$$\begin{array}{ccccc} x & \xleftarrow{\quad} & z & \xrightarrow{\quad} & y \\ a \downarrow & & b \downarrow & & c \downarrow \\ x' & \xleftarrow{\quad} & z' & \xrightarrow{\quad} & y' \end{array}$$

where a, b and c are morphisms in $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$. What we need to prove is the induced morphism $a \sqcup_b c: x \sqcup_z y \rightarrow x' \sqcup_{y'} z'$ is also in $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$. By applying the functor \mathcal{L}_* to the diagram above, it turns out that $\mathcal{L}_*(a \sqcup_b c) = \mathcal{L}_*(a) \sqcup_{\mathcal{L}_*(b)} \mathcal{L}_*(c)$ is an isomorphism by exactness of \mathcal{L}_* . Thus $a \sqcup_b c: x \sqcup_z y \rightarrow x' \sqcup_{y'} z'$ is in $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$.

(9) Let $a: x \rightarrow y$ be a morphism in $wF_n\mathcal{C}$. Then by (3), for each integer $0 \leq k \leq n$, there are a morphism $b_k: y_k \rightarrow x_k$ and elements s_k, t_k and u_k in \mathfrak{S} such that $a_k b_k t_k = s_k t_k \text{id}_{y_k}$ and $b_k a_k u_k =$

id_{x_k} . We set $b'_k := \left(\prod_{\substack{i=0 \\ i \neq k}}^n s_i \right) \left(\prod_{i=0}^n (u_i t_i) \right) b_k$ and $c := \prod_{i=0}^n (s_i t_i u_i)$. Then we have the equalities $a_k b'_k =$

$c \text{id}_{y_k}$ and $b'_k a_k = c \text{id}_{x_k}$. Therefore for each $0 \leq k \leq n-1$, we have the equalities

$$i_k^x b'_k a_k = c i_k^x = b'_{k+1} a_{k+1} i_k^x = b'_{k+1} i_k^y a_k.$$

Moreover since $\mathcal{L}_*(a_k)$ is an isomorphism in $\mathcal{M}_{\mathfrak{S}^{-1}A}^{\mathcal{J}}$, the diagram below is commutative:

$$\begin{array}{ccc} \mathcal{L}_*(y_k) & \xrightarrow{\mathcal{L}_*(i_k^y)} & \mathcal{L}_*(y_{k+1}) \\ \mathcal{L}_*(b'_k) \downarrow & & \downarrow \mathcal{L}_*(b'_{k+1}) \\ \mathcal{L}_*(x_k) & \xrightarrow{\mathcal{L}_*(i_k^x)} & \mathcal{L}_*(x_{k+1}). \end{array}$$

Hence by (1), there is an element v_k in \mathfrak{S} such that $v_k b'_{k+1} i_k^y = i_k^x v_k b'_k$. We set $v := \prod_{i=0}^{n-1} v_i$ and for each integer $0 \leq k \leq n$, we set $b''_k := v b'_k$. Then for any integer k , we have the equalities $i_k^x b''_k = b''_{k+1} i_k^y$. Namely b'' is a morphism from y to x in $F_n \mathcal{C}$ and we have the equalities $ab'' = cv \text{id}_y$ and $b''a = cv \text{id}_x$. By applying (3) to the morphisms a and b'' and the element cv of \mathfrak{S} , it turns out that a is in $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$. Conversely we can show that a morphism in $\text{Isom}_{\mathfrak{S}, F_n \mathcal{C}}$ is contained in $w F_n \mathcal{C}$ by utilizing (3) again. We complete the proof.

(11) Obviously assumption says that the canonical functor $w^{-1} S_n \mathcal{C} \rightarrow S_n w^{-1} \mathcal{C}$ is essentially surjective. Then by (10), it turns out that the functor is an equivalence of categories $w^{-1} S_n \mathcal{C} \xrightarrow{\sim} S_n w^{-1} \mathcal{C}$.

(12) By (11), for any non-negative integer n , the canonical functor $w^{-1} S_n \mathcal{C} \rightarrow S_n w^{-1} \mathcal{C}$ is an equivalence of categories. Hence by Corollary 3.4, we obtain the result.

(13) First notice that g is a cofibration in $\mathcal{M}_A^{\mathcal{J}}$. Let us consider the commutative diagram in $\mathcal{M}_A^{\mathcal{J}}$ below:

$$\begin{array}{ccccc} x & \rightharpoonup & y & \twoheadrightarrow & y/x \\ \parallel & & \downarrow & & \downarrow \\ x & \rightharpoonup & z & \twoheadrightarrow & z/x \\ \downarrow & & \parallel & & \downarrow \\ y & \rightharpoonup & z & \twoheadrightarrow & z/y. \end{array}$$

Here the sequence $x \rightharpoonup y \twoheadrightarrow y/x$ is a cofibration sequence in $\mathcal{M}_A^{\mathcal{J}}$. On the other hand, by assumption, $y/x \xrightarrow{\sim} \ker(z/x \twoheadrightarrow z/y)$ is isomorphic to an object in \mathcal{C} . Since the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{M}_A^{\mathcal{J}}$ reflects exactness, g is a cofibration in \mathcal{C} .

(14) What we need to show is that for a morphism $f: x \rightarrow y$ in \bar{w} and a cofibration $g: z \rightarrow y$ in \mathcal{C} , there exists an object u in \mathcal{C} and a morphism $s': u \rightarrow z$ in \bar{w} and a cofibration $h': u \rightarrow x$ in \mathcal{C} such that $gs' = fh'$. In this case, by (4), there are a morphism $h: z \rightarrow x$ and an element s of \mathfrak{S} such that $gs = fh$. Then since f and g are monomorphisms in \mathcal{C} , we have equalities $\ker s = \ker gs = \ker fh = \ker h$. Thus the morphisms $s: z \rightarrow z$ and $h: z \rightarrow x$ induce the morphisms $\bar{s}: z/\ker s \rightarrow z$ and $\bar{h}: z/\ker s \rightarrow x$ which makes the diagram below commutative:

$$\begin{array}{ccc} z/\ker s & \xrightarrow{\bar{s}} & z \\ \bar{h} \downarrow & & \downarrow g \\ x & \xrightarrow{f} & y. \end{array}$$

By assumption, the sequence $z/\ker s \rightharpoonup z \twoheadrightarrow \text{Coker } s$ is a cofibration sequence in \mathcal{C} . Hence \bar{s} is a morphism in \bar{w} . Then by the equality $g\bar{s} = f\bar{h}$ and (13), \bar{h} is also a cofibration in \mathcal{C} . We complete the proof. \square

4.5. Corollary. Let \mathcal{C} be a full subcategory of $\mathcal{M}_A^{\mathcal{J}}$ and we write w for $\text{Isom}_{\mathfrak{S}, \mathcal{C}}$. We assume the following conditions:

- (1) \mathcal{C} and $w^{-1} \mathcal{C}$ are categories with cofibrations such that the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{M}_A^{\mathcal{J}}$ and the canonical functor $w^{-1} \mathcal{C} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}^{\mathcal{J}}$ induced from the base change functor $- \otimes_A \mathfrak{S}^{-1} A: \mathcal{C} \rightarrow \mathcal{M}_{\mathfrak{S}^{-1} A}^{\mathcal{J}}$ are exact and reflect exactness.
- (2) \mathcal{C} is closed under taking kernels of the morphisms in \mathcal{C} which are epimorphisms in $\mathcal{M}_A^{\mathcal{J}}$.
- (3) For any element s of \mathfrak{S} and any object x in \mathcal{C} , the objects $\text{Coim}(s: x \rightarrow x)$ and $\text{Coker}(s: x \rightarrow x)$ are also in \mathcal{C} .
- (4) For any non-negative integer n , and any object x in $S_n w^{-1} \mathcal{C}$, there exists an object y in $S_n \mathcal{C}$ and a morphism $y \rightarrow x$ in w .
- (5) The set of trivial cofibrations $\bar{w} := w \cap \text{Cof } \mathcal{C}$ is right cofinal in w .

Then the inclusion functor $\mathcal{C}^w \hookrightarrow \mathcal{C}$ and the canonical functor $\mathcal{C} \rightarrow w^{-1}\mathcal{C}$ induce a fibration sequence up to homotopy

$$iS.\mathcal{C}^w \rightarrow iS.\mathcal{C} \rightarrow iS.w^{-1}\mathcal{C}.$$

Proof. We will apply Corollary 3.5 to the sequence $iS.\mathcal{C}^w \rightarrow iS.\mathcal{C} \rightarrow iS.w^{-1}\mathcal{C}$. The assumptions in the Corollary follow from Definition-Corollary 4.4. \square

Finally we give an application of Proposition 3.6 to categories of diagrams of modules.

4.6. Corollary. *Let \mathcal{C} be a full subcategory of $\mathcal{M}_A^{\mathcal{J}}$ and let \mathcal{D} be a full subcategory of \mathcal{C} . We write $w_{\mathcal{C}}$ and $w_{\mathcal{D}}$ for $\text{Isom}_{\mathfrak{S},\mathcal{C}}$ and $\text{Isom}_{\mathfrak{S},\mathcal{D}}$ respectively. Assume that \mathcal{C} and \mathcal{D} are categories with cofibrations such that the inclusion functors $\mathcal{D} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}_A^{\mathcal{J}}$ are exact and reflect exactness. Suppose that the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ satisfies the following two conditions.*

- (1) *For any object x in \mathcal{C} , there are an object y in \mathcal{D} and a morphism $h: y \rightarrow x$ in $w_{\mathcal{C}}$.*
- (2) *For any cofibration $i: x \rightarrow y$ in \mathcal{C} and any morphism $a: y' \rightarrow y$ in $w_{\mathcal{C}}$ with y' in \mathcal{D} , there are a cofibration $i': x' \rightarrow y'$ in \mathcal{D} and a morphism $b: x' \rightarrow x$ in $w_{\mathcal{C}}$ such that $ib = ai'$.*

$$\begin{array}{ccc} x' & \xrightarrow{b \in w_{\mathcal{C}}} & x \\ i' \downarrow & & \downarrow i \\ y' & \xrightarrow{a \in w_{\mathcal{C}}} & y \end{array}$$

Then the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ induces a homotopy equivalence $w_{\mathcal{D}}S.\mathcal{D} \rightarrow w_{\mathcal{C}}S.\mathcal{C}$.

Proof. We will apply Proposition 3.6 to the inclusion functor $(\mathcal{D}, w_{\mathcal{D}}) \rightarrow (\mathcal{C}, w_{\mathcal{C}})$. For simplicity we write w for both $w_{\mathcal{C}}$ and $w_{\mathcal{D}}$. Let n be a non-negative integer. Since $S_n\mathcal{C}$ is equivalent to $F_{n-1}\mathcal{C}$ and $F_{n-1}\mathcal{C}$ is a full subcategory of $\mathcal{C}^{[n-1]}$, $wS_n\mathcal{C}$ is right localizing in $S_n\mathcal{C}$ by Definition-Corollary 4.4 (5). Similarly $wS_n\mathcal{D}$ is right localizing in $S_n\mathcal{D}$. We will show that the inclusion functor $wS_n\mathcal{D} \hookrightarrow wS_n\mathcal{C}$ induces an equivalence of categories $\iota_*: w^{-1}S_n\mathcal{D} \xrightarrow{\sim} w^{-1}S_n\mathcal{C}$. To prove that ι_* is essentially surjective, we claim the following assertion.

Claim. For any object x in $F_n\mathcal{C}$ and a morphism $s: y \rightarrow x_n$ in $w_{\mathcal{C}}$, there is an object x' in $F_n\mathcal{D}$ and a morphism $a: x' \rightarrow x$ in $wF_n\mathcal{C}$ such that $x'_n = y$ and $a_n = s$.

Proof of claim. We proceed by induction on n . If $n = 1$, then assertion is trivial, and if $n = 2$, assertion is just assumption (2). Suppose that assertion is true for $n = n'$ and we will consider assertion for $n = n' + 1$. Let x be an object in $F_{n'+1}\mathcal{C}$ and let $s: y \rightarrow x_{n'+1}$ be a morphism in $w_{\mathcal{C}}$. Then by assumption (2), there are a cofibration $i': y' \rightarrow y$ and a morphism $t: y' \rightarrow x_{n'}$ in w such that $i_{n'}^x t = si'$. By inductive hypothesis, there is an object x'' in $F_{n'}\mathcal{D}$ and a morphism $u: x'' \rightarrow j_*x$ in $wF_{n'}\mathcal{C}$ such that $x''_{n'} = y'$ and $u_{n'} = t$. Then we define x' and $a': x' \rightarrow x$ to be an object in $F_{n'+1}\mathcal{D}$ and a morphism in $wF_{n'+1}\mathcal{D}$ as follows.

$$x'_k = \begin{cases} x''_k & \text{if } k \leq n' \\ y & \text{if } k = n' + 1 \end{cases}, \quad i_k^{x'} = \begin{cases} i_k^{x''} & \text{if } k \leq n' - 1 \\ i' & \text{if } k = n' \end{cases}, \quad a_k = \begin{cases} u_k & \text{if } k \leq n' \\ s & \text{if } k = n' + 1. \end{cases}$$

We can prove that x' and a are what we want. Thus we complete the proof. \square

In particular, it turns out that ι_* is essentially surjective. To prove ι_* is fully faithful, we will utilize Lemma 9.1 in [Kel96]. To apply the lemma, what we need to prove is that for any morphism $s: x \rightarrow x'$ in $w_{\mathcal{C}}$ with x' in \mathcal{D} , there are an object x'' in \mathcal{D} and a morphism $m: x'' \rightarrow x$ such that $sm: x'' \rightarrow x'$ is in $w_{\mathcal{D}}$. This condition follows from Definition-Corollary 4.4 (3). Hence the functor $\iota_*: w^{-1}S_n\mathcal{D} \xrightarrow{\sim} w^{-1}S_n\mathcal{C}$ is an equivalence of categories. Then by Proposition 3.6, the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ induces a homotopy equivalence $wS.\mathcal{D} \rightarrow wS.\mathcal{C}$. \square

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